Flower Constellation Set Theory Part II:
Secondary Paths and Equivalency

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Abstract—In our previous research, the Flower Constellation Set theory was introduced but specific details left out. In this work, the particular phasing theory that we have adopted is discussed in full. As a consequence of this choice of parametrization, a new class of orbit theory has emerged: secondary paths. The theory of secondary paths is developed and proved in this work. Examples of secondary paths are presented and discussed. Furthermore, we discuss the equivalency of Flower Constellations and resolve how certain disparate choices of integer parameters can generate identical satellite distributions.

Index Terms—Flower Constellations, Repeating space track, Compatible orbits, Large arrays.

INTRODUCTION

Flower Constellations present an interesting and unique mathematical theory for the design of constellations and formations of satellites. As discussed in a companion paper, to establish a Flower Constellation (FC), one must specify the semi-major axis and eccentricity through a choice of parameters \((N_p, N_d, h_p)\), the orbit inclination, and the argument of perigee, which are all common values for each satellite belonging to that Flower Constellation. The remaining two parameters, the right ascension of the ascending node (RAAN) and the mean anomaly, must be determined through a phasing rule for each satellite.

In our previous work for the Flower Constellation Set theory, we developed a functional relationship between the right ascension of the ascending node and the mean anomaly of a given satellite such that its relative trajectory in an arbitrary rotating reference frame will precisely coincide with the relative trajectory of another satellite in that frame. We did not elaborate on a specific choice of this functional relationship because the functional choice is largely an arbitrary choice. However, we made a specific choice for the parametrization, and a new class of orbit theory has emerged, secondary paths (SPs).

Symmetric Schemes

The symmetric phasing scheme that has been adopted for use in the Flower Constellations, characterized by \(N_s\) satellites, is obtained by the phasing rule

\[
\Omega_{k+1} = \Omega_k - 2\pi \frac{F_n}{F_d}
\]

(1)

\[
M_{k+1}(0) = M_k(0) + 2\pi \frac{\omega_k \Omega_0}{\omega_0 + \Omega_0}
\]

(2)

where \(k = 1, \ldots, N_s - 1\), \(\omega_0\) is the angular velocity of the arbitrary rotating reference frame, \(\Omega\) and \(M_0\) are perturbations to the RAAN and mean anomaly angles, \(F_n\) and \(F_d\) are two integer parameters that can be freely chosen provided that \(F_n \in \mathbb{Z}\) and \(F_d \in \mathbb{N}\), and where \(M_1(0)\) and \(\Omega_1\) (which are assigned) dictate the initial position of the first satellite and the angular shifting of the relative orbit, respectively.\(^1\) Flower Constellations have an axis of symmetry that is coincident with the direction of the total angular moment vector of all the satellite orbits that constitute the constellation.

Please pay attention to the signs of Eqs. (1) and (2). They are opposite of the sign convention used in the companion paper. Either form of the equations is correct, and the resulting analysis from using the equations is identical.

Equation (2) has a more simplified form that allows for more extensive analysis. If perturbations are ignored, then the most simplified version of the phasing relationships are given by

\[
\Delta \Omega = -2\pi \frac{F_n}{F_d}
\]

(3)

\[
\Delta M = 2\pi \frac{F_n N_p}{F_d N_d} = -\Delta \Omega \frac{N_p}{N_d} = -\Delta \Omega \frac{1}{\tau}
\]

(4)

where \(\tau \equiv \frac{N_d}{N_p}\).

Note that \(\Delta \Omega\) and \(\Delta M\) as expressed in Equation (3) and Equation (4) are both rational, constructible numbers. It becomes clear here that the maximum number of satellites in a given Flower Constellation is

\[
N_s^* = F_d N_d
\]

(5)

and that the sequence of right ascension of the ascending node values will repeat \(N_d\) times. This is due to the fact that a single RAAN and mean anomaly value will be assigned in sequence to each satellite in a unique pairing. Since the mean anomaly steps by \(F_d N_d\), then it will take a total of \(F_d N_d\) steps in order to complete the assignments. It follows from \(\Delta \Omega F_d N_d\) that the unique values of \(\Delta \Omega\) will repeat \(N_d\) times. Note that Equation (5) is an upper bound on the number of satellites. One is not required to completely fill out a Flower Constellation but rather can selectively choose where to place satellites once the phasing scheme has been established. Therefore,

\[
N_s \leq F_d N_d
\]

(6)

\(^1\)Note that this choice can be somewhat limiting in that irrational numbers are excluded. To avoid this limitation, \(F_n/F_d\) should be substituted by a single decimal parameter, \(F\), where \(F \in \mathbb{R}\). With this choice, all the currently known possible types of symmetric phasing are encompassed. As more investigation is completed, additional symmetric schemes may become apparent.
Restricted Schemes

Consider now that, for some mission design reason, the RAAN angle is constrained to fall within a certain range. These constellations are built with orbit node lines uniformly distributed in a limited range \([\Omega_1, \Omega_2]\) instead of \([0, 2\pi]\). In this case, the phasing rules given in Eqs. (1) and (2) are specialized as follows

\[
\Omega_{k+1} = \Omega_k - (|\Omega_2^k - \Omega_1^k|) \frac{F_i}{F_d} = \Delta \Omega_k \frac{F_i}{F_d} \tag{7}
\]

\[
M_{k+1}(0) = M_k(0) + \Delta \Omega_k \frac{n + M_0}{\omega_s + \Omega} \tag{8}
\]

with \(\Omega_1 = \Omega_2^k\) and \(M_1 = M_1(0)\). Because the RAAN angle is decremented, one must remember to set \(\Omega_1\) to the maximum value in the range of allowable values for the restricted RAAN scheme.

**Non-symmetric Schemes**

Building upon the concept of the restricted phasing scheme, the phasing relationships can also be expressed as

\[
\Omega_{k+1} = \Omega_k - \Delta \Omega_k \tag{9}
\]

\[
M_{k+1}(0) = M_k(0) + \Delta \Omega_k \frac{n + M_0}{\omega_s + \Omega} \tag{10}
\]

with \(\Omega_1\) and \(M_1(0)\) assigned. This implies that the difference in node value between any two satellites in the placement sequence can be arbitrary provided that the mean anomaly is selected appropriately.

Incomplete Schemes

Based upon the developments of the previous section, it is clear that changing the number of satellites in the constellation does not have any dramatic effect on the overall dynamics of the Flower Constellation, which, in turn, is dictated by the overall structure (parameters \(N_d\) and \(N_s\)) and the phasing rules (parameters \(F_i\) and \(F_d\)). Once a desired dynamic is achieved by a proper choice of the constellation parameters, then it becomes an easy task to find out the minimum number of satellites required to accomplish a specific mission objective. If it is desirable to remove a portion of the satellites but maintain the overall dynamics, one must generate the Flower Constellation as if the complete number of satellites were going to be placed and then selectively remove the undesirable number of satellites. That is to say, once the orbit elements have been generated for all possible satellites, the mission designer can selectively choose sets of parameters from the list. This procedure is necessary because of the way that the phasing rules are mathematically constructed. Doing this leads to an incomplete Flower Constellation.

SECONDARY CLOSED PATHS - EXISTENCE AND UNIQUENESS

In the previous section, a method for generating a single closed path - the relative path - is described. All the satellites belonging to a particular Flower Constellation belong to that single relative path. In order for this to occur, each satellite is assigned a unique pairing of RAAN and mean anomaly angles \((\Omega_i, M_i(0))\) while their remaining orbit parameters are identical \((a, e, i, and \omega)\). Ignoring perturbations, Equation (3) and Equation (4) define the allowable values for these pairs.

Considering Equation (4), the maximum value of \(M(0)\) is

\[
M_f(0) = 2\pi F\omega N_p.
\]

One can also see that there are a maximum of \(F\omega N_p\) unique mean anomaly angles. This is graphically illustrated in Figure 1. Furthermore, examining Equation (3), one can see that the maximum value of RAAN is \(2\pi F\omega\). However, since there are more available mean anomaly angles to assign, the RAAN must cycle until a value has been assigned to each corresponding value of the mean anomaly. Therefore, the final value of RAAN will be

\[
\Omega_f = \Delta \Omega F\omega N_d = -2\pi F\omega N_d.
\]

Consider the possibility that the pairs of RAAN and mean anomaly angles will repeat before \((\Omega_f, M_f(0))\) is reached. In this case, there exists only a subset of angle pairs that are unique. This subset of unique angle pairs is what is termed a secondary closed path (SCP). If one were to continue placing satellites in the standard fashion, they would be placed physically on top of one another. While this might be a mathematical possibility, it is a physically unrealizable condition! Thus, the secondary closed path is a unique pattern of satellites that lies on top of the original relative path. A satellite belonging to a secondary closed path also belongs to the original relative path.

A question is now posed. What values of the Flower Constellation design parameters will cause a secondary closed path to occur?

**Theorem 1:** For \(N_p\) sufficiently large, a symmetric Flower Constellation secondary closed path occurs when \(N_d = 1\) or \(F_n = k \omega N_d\) for \(k \in \mathbb{N}\).

**Remark 2 (Theorem 1):** \(N_p\) is the number of “petals” in the relative path of a Flower Constellation. In order for a secondary closed path to be distinguishable, the number of petals must be large enough that they overlap sufficiently to allow for the closed path to be obvious. The choice of \(N_p\) and the relative merit of “sufficiently large” is left up to the mission designer.

**Proof:** In order for the \((\ Omega_i, M_i(0))\) pairs to repeat and form a unique subset, the mean anomaly angle must be an integer multiple of \(2\pi\) less than \(M_f(0)\). Assume that there are a maximum of \(N_p\) satellites in this subset. This leads to

\[
\Delta M m = 2\pi N_p
\]

**Fig. 1.** Comparison of the sequence of allowable values for the RAAN and mean anomaly angles. Each mark on the number line corresponds to a unique pair of RAAN and mean anomaly angles that specifies a location for the \(i\)-th satellite to be placed in an orbit. All other orbit parameters have been specified by the design of the Flower Constellation.
Likewise, for the RAAN, one can write
\[ \Delta \Omega = 2\pi n_2 \] (12)
where \( m, n_1 \) and \( n_2 \) are unknown integers.

This can be written out in equation form as
\[ -2\pi m \frac{F_n}{F_d} = 2\pi n_1 \] (13)
\[ 2\pi m \frac{N_p}{F_d} = 2\pi n_2 \] (14)
which reduces to
\[ -m \frac{F_n}{F_d} = n_1 \] (15)
\[ m \frac{F_n}{F_d} = n_2 \] (16)

It has already been established that \( F_n \) and \( F_d \) must be relatively prime for a unique Flower Constellation as must be \( N_p \) and \( N_d \). Examining Equation (15), one can immediately say by the division lemma, one can easily see that, since \( F_n \parallel F_d \), \( m \) for \( n_1 \) to be an integer (i.e. \( m = jF_d \) for \( j \in \mathbb{N} \)). The smallest integer value of \( F_d \) that divides \( m \) is required for a unique base pattern due to the simple fact that multiples of \( m \) will only result in multiples of the base secondary closed path. Therefore, \( m = F_d \), and consequently, \( n_1 = -F_n \).

Equation (16) can be analyzed in a similar way. Here, \( (F_d N_d) \parallel (m F_d N_p) \). However, since it was just established that \( m = F_d \), this reduces to \( N_d \parallel (F_d N_p) \). From this condition and the division lemma, one can easily see that, since \( N_p \parallel N_d \), either \( N_d = 1 \) or \( N_d \parallel F_n \). In other words, a unique secondary closed path is formed when \( F_n \) is an integer multiple of \( N_d \) (i.e. \( F_n = kN_d \) for \( k \in \mathbb{N} \)). When \( k > 1 \), then a unique set of multiple base paths will form. These requirements are graphically depicted in Figure 2. Note that when \( F_n = N_d \), then \( n_1 = -F_n = -N_d \) and \( n_2 = N_p \). Likewise, when \( N_d = 1 \), \( n_2 = F_n N_p \).

**Corollary 3 (Theorem 1):** For \( N_p \) sufficiently large, a Flower Constellation secondary closed path can also occur when \( F_n \parallel N_d \) (i.e. \( N_d = rF_n \) for \( r \in \mathbb{N} \)).

**Proof:** In Theorem 1, it was shown that a secondary closed path occurs when \( N_d \parallel F_n \), which is equivalent to saying that \( F_n \) is an integer multiple of \( N_d \). If one rearranges this requirement to show that \( F_n = kN_d \), one can see immediately that \( N_d \) can only be an integer when \( k \parallel F_n \). Therefore, a secondary closed path can also occur when \( N_d = rF_n \) where \( r \in \mathbb{N} \).

One can see clearly now that the secondary closed path subset of RAAN and mean anomaly pairs will repeat after \( F_d \) pairs. Thus, the maximum number of satellites in a secondary closed path is
\[ N_s = F_d \] (17)

Note that \( N_s \leq F_d \) with \( N_s = F_d \) in order to completely visualize the entire secondary closed path. At this point, not much has been said about the actual values of \( N_p, N_d, \) or \( F_d \). From previous developments, \( N_p \) and \( N_d \) determine the anomalistic orbit period of the satellites. Thus, these values would generally be established by mission design requirements. \( F_d \), however, has no such constraints and, in general, can be any non-zero positive integer (i.e. \( F_d \in \mathbb{N} \)). Given an infinity of choices for the value of \( F_d \), we can choose \( F_d \) in a specific way that allows one to predict the resultant secondary path shape.

**Theorem 4:** For \( N_p \) sufficiently large, and assuming that Theorem 1 has been satisfied, the phasing denominator can be expressed as \( F_d = AN_d + BN_p \) where \( F_d \in \mathbb{N}, N_p \) and \( N_d \) are specified according to Flower Constellation theory, and given arbitrary \( A,B \in \mathbb{Z} \) such that \( A \) and \( B \) have the physical meaning of the integer number of times the mean anomaly and the RAAN are divisible by \( 2\pi \).

**Proof:** Examine Equation (3) and Equation (4) now written for the specific case of the secondary closed path:
\[ \Delta \Omega = \frac{2\pi N_d}{F_d} \] (18)
\[ \Delta M_0 = \frac{2\pi N_p}{F_d} \] (19)

In order to determine the form of \( F_d \), one must demonstrate that the form of either Equation (18) or Equation (19) can be constructed. Since Equation (18) and Equation (19) are dependent, we only have to prove one of the equations. First consider
\[ \Delta M_0 \equiv C \mod 2\pi \] (20)
\[ \Delta \Omega \equiv D \mod 2\pi \] (21)

where the symbol \( \equiv \) means congruent. According to the definition of congruency,
\[ a \equiv c \mod b \iff b \mid (a - c). \] (22)
Therefore, one can write that
\[ 2\pi | (\Delta M_0 - C) = A \]
\[ 2\pi | (\Delta \Omega - D) = B \]

where A and B are integers (i.e., \( A, B \in \mathbb{Z} \)) and C and D are real numbers (i.e., \( C, D \in \mathbb{R} \)).

From Equation (4), one can express \( \Delta M \) in terms of \( \Delta \Omega \). This leads to
\[ 2\pi | \left( -\Delta \Omega \frac{N_p}{N_d} - C \right) = A \]
\[ 2\pi | (\Delta \Omega - D) = B \]

which can also be written as
\[ 2\pi A = \left( -\Delta \Omega \frac{N_p}{N_d} - C \right) \]
\[ 2\pi B = (\Delta \Omega - D) \]

Solving for \( 2\pi \) and then equating to the two equations results in
\[ \frac{A}{B} (\Delta \Omega - D) = -\Delta \Omega \frac{N_p}{N_d} - C \]

Collecting terms,
\[ \Delta \Omega \left( A + \frac{N_p}{N_d} \right) = \frac{A}{B} D - C \]

Finding the common denominator of the term in parentheses,
\[ \Delta \Omega \left( \frac{AN_d + BN_p}{BN_d} \right) = \frac{A}{B} D - C \]

Multiplying through,
\[ \Delta \Omega (AN_d + BN_p) = ADN_d - BCN_d = (AD - BC)N_d \]

which leads to
\[ \Delta \Omega = \frac{(AD - BC)N_d}{AN_d + BN_p} = \frac{2\pi N_d}{F_d} \]

where it is now clear comparing Equation (33) to Equation (18) that \( F_d = AN_d + BN_p \) and A and B represent the integer number of times that the RAAN and the mean anomaly are divisible by \( 2\pi \).

What is most interesting is that the values selected for A and B have physical meaning. One can think of B as the number of times the secondary closed path will intersect itself or twist over onto itself. A is the whole number of times that the node angle will be swept through \( 2\pi \). One can see this clearly by plotting the RAAN (\( \Omega \)) versus the mean anomaly (\( M \)) in cartesian coordinates for various values of A and B. Figure 3 shows 6 cases where \((A, B) \in \{(0, 3) \ (3, 0) \ (2, 3) \ (3, 2) \ (-1, 3) \ (3, -1) \}\).  

For an arbitrary choice of \( F_d \), the plot of RAAN and mean anomaly can have a sparse scatter plot appearance. This is in large part due to the fact that the maximum number of satellites is controlled by \( F_d \) (assuming \( N_d \) is a defined mission parameter). However, when \( F_d \) is chosen to have the form given in Theorem 4, then the scattered points will coalesce into distinct bands. Therefore, it is important to consider that while Theorem 1 guarantees a secondary closed path will exist, Theorem 4 determines in large part if the secondary closed path is distinguishable to the human eye. Furthermore, we can conclude that the categories of Flower Constellations discussed in our previous work are merely human interpretations of a mathematical phenomena.

**Secondary Open Paths**

Referring back to Equation (4), recall that the ratio of \( N_p \) and \( N_d \) is defined as \( \tau \). Keeping the values of \( N_p \) and \( N_d \) in proportion to one another is important for realizable constellations and formations to be generated. For a given value of \( N_d \), as \( N_p \to \infty \), \( \tau \to 0 \), the anomalistic orbit period also goes to zero (which is physically unrealizable). Conversely, for a given value of \( N_p \), as \( N_d \to \infty \), \( \tau \to \infty \), the anomalistic orbit period goes to infinity. This is also generally unacceptable. Therefore, even though \( N_p \) and \( N_d \) may individually be quite large, provided that the overall ratio, \( \tau \) is kept in a reasonable range, the resultant Flower Constellation will be plausible.

Note that a lower bound for \( \tau \) can be found as a function of some minimum allowable perigee altitude. This lower bound...
can be expressed as

$$\tau \geq \omega \sqrt{\frac{(R_p + h_{\text{min}})^3}{\mu_p}}$$

(34)

where $h_{\text{min}}$ is the minimum altitude above the planet, $R_p$ is the mean planet radius, and $\mu_p$ is the gravitational parameter of the planet under consideration. Table I shows the resultant minimum value for $\tau$ for some specific values of the minimum altitude $h_{\text{min}}$. Here we have written the equations in a general way to reference any planet to remind the reader that the Flower Constellations can be designed with respect to any planet or rotating reference system.

Because Flower Constellations are geometrically symmetrical about the axis of symmetry, as the number of petals increases while the number of days to repeat the pattern stays the same, the petals begin to overlap each other. As $N_p$ grows large, the relative orbit begins to look less like distinct flower petals and more like a surface. Figure 6 shows two cases to illustrate this point. The first case shows a $N_p = 3, N_d = 1$ Flower Constellation ($\tau = 0.3333$) while the second case has $N_p = 769, N_d = 257$ ($\tau = 0.3342$).

Now, remove the relative orbits that are depicted in Figure 5 and, instead, increase the number of satellites to fill out the pattern. When one views both of these cases from the North pole of the Earth without the relative orbits (Figure 6), one will see something rather unexpected. Both constellations look almost identical.

In fact, there is only one subtle difference between the two patterns of satellites. In Case 1, the pattern fits exactly on the relative path and, thus, is a closed path (this is the trivial secondary closed path case). In Case 2, there are two "paths" for the satellites to follow. The first "path" is the relative
orbit depicted in Figure 5. The secondary path is generated by the pattern that you see in Figure 6. The secondary path, for this particular case, is open ended. That is, if you follow a particular satellite from the starting point to the end point of the pattern, those points do not coincide. As the number of petals is decreased (while maintaining the appropriate \( \tau \) ratio), this disconnection will become more and more pronounced.

Note that, in general, secondary paths can be closed or open. Part of the constellation design process will focus on selecting patterns of satellites that are either on closed or open secondary paths. Secondary open paths are formed when \( \tau \) is perturbed slightly and all other design parameters remain constant.

**Restricted Phasing Secondary Paths**

The orbit distribution as provided by Eqs. (3) and (4) is evenly distributed about a complete \( 2\pi \) rotation. When the orbit node lines are restricted to lie within a particular range, then the existence of secondary paths must be revisited. As will be seen, the mathematical possibilities become very limited if the orbits are not arrayed about \( 2\pi \).

**Theorem 5:** For \( N_p \) sufficiently large and in the special case when one restricts the right ascension of the ascending node of each orbit to lie within the range \( [\Omega_1^*, \Omega_1^*] \), a Flower Constellation secondary closed path occurs when \( F_d = N_o, N_d = F_o = 1 \), and \( N_p = kN_\Omega \forall \, k \in \mathbb{N} \) where \( N_o \) is the number of orbit planes and \( \Delta\Omega_R = [\Omega_1^* - \Omega_1^*] \) is chosen to have the functional form \( \Delta\Omega_R = 2\pi/N_\Omega, \, N_\Omega \in \mathbb{N} \).

**Remark 6:** Note that this choice of functional form for \( \Delta\Omega_R \) restricts the values that \( \Delta\Omega_R \) can be assigned.

**Proof:** In our previous work, we showed that the relationship between RAAN and the mean anomaly is

\[
\Delta M = \frac{-\Delta\Omega}{N_d} = 2\pi \frac{F_o N_p}{F_d N_d} \quad (35)
\]

where

\[
\Delta\Omega = -2\pi \frac{F_o}{F_d} \quad (36)
\]

This provides for a Flower Constellation that has orbits whose RAAN are evenly arrayed about \( 2\pi \). However, we may desire to have orbits that are restricted to be arrayed within a smaller range of RAAN values, \( \Delta\Omega_R \). In this case, while Equation (35) remains the same, Equation (36) becomes

\[
\Delta\Omega = \frac{-\Delta\Omega_R}{N_d} = 2\pi \frac{F_o}{F_d} \quad (37)
\]

where

\[
\Delta\Omega_R = \frac{2\pi}{N_\Omega} \quad (38)
\]

For example, if one desires to restrict the RAAN to a span of 60\(^\circ\), then \( N_\Omega = 6 \). Now, let’s re-write Equation (37) as

\[
\Delta\Omega = \frac{-\Delta\Omega_R}{F_d} = 2\pi \frac{|F_o \Omega_1^* - F_o \Omega_1^*|}{F_d} \quad (39)
\]

while Equation (35) becomes

\[
\Delta M = \frac{-\Delta\Omega N_p}{N_d} = 2\pi \frac{F_o N_p}{N_\Omega F_d N_d} \quad (40)
\]

Notice here that, for \( F_o > 1 \), the value of \( \Omega_1 \) is not \( \Omega_1^* \) as we had originally intended. Therefore, if one desires to restrict all orbits to \( \Omega_1^* < i < \Omega_2^* \equiv N_o \), then \( F_o \equiv 1 \).

\[
\Delta\Omega = -\frac{\Delta\Omega_R}{F_d} = -\frac{2\pi}{N_\Omega F_d} \quad (41)
\]

and

\[
\Delta M = \frac{2\pi N_p}{N_\Omega F_d N_d} \quad (42)
\]

Pay close attention to Equation (41). There are two equally valid representations depicted. However, we must choose

\[
\Delta\Omega = -\frac{\Delta\Omega_R}{F_d} \quad (43)
\]

and remember to set \( \Omega_{i+1} = \Omega_1^* + \Delta\Omega \) when \( \Omega_2^* = \Omega_1^* \). If we had chosen the latter representation, then we would have ended up with an *Flower Constellation* arrayed about \( 2\pi \) again.

Now, the question becomes, what values of the remaining Flower Constellation parameters will result in a secondary closed path on a restricted RAAN set. As discussed in earlier, SCP’s form when a unique subset of RAAN and mean anomaly pairs repeat an integer number of times within the allowable range of discrete mean anomaly values. However, there are some subtle differences with the restricted RAAN case. In order for the \( (\Omega, M(0)) \) pairs to repeat and form a unique subset, the mean anomaly angle must be an integer multiple of \( 2\pi \) less than \( M_f(0) \). Likewise, since we have restricted the RAAN, the RAAN angle must be an integer multiple of \( \Delta\Omega_R \). Assume that there is a maximum of \( N_s^* \) satellites in this subset. This leads to

\[
-m \Delta\Omega = \Delta\Omega_R n_1 \quad \text{and} \quad m \Delta M = 2\pi n_2 \quad (44)
\]

Substituting in the relationships for \( \Delta\Omega \) and \( \Delta M \) from the above equations, we obtain

\[
-m \frac{\Delta\Omega_R}{F_d} = \Delta\Omega_R n_1 \quad (45)
\]

and

\[
2\pi m \frac{N_p}{N_\Omega F_d N_d} = 2\pi n_2 \quad (46)
\]

Examining Equation (45) in detail, we see that

\[
-\frac{m}{F_d} = n_1 \quad (47)
\]

which implies that \( F_d = m n_1 \) to be an integer. Furthermore, for a base SCP, \( F_d = m \). Substituting this result into Equation (46), we get the following:

\[
N_p = N_\Omega N_d n_2 \quad (48)
\]

It follows that, for \( n_2 \) to be an integer, \( N_\Omega N_d \mid N_p \). It has been well established that \( N_d \perp N_p \). That is to say, \( N_d \) and \( N_p \) are relatively prime. Therefore, the only possible solution that will result in \( n_2 \) being an integer is the case when \( N_d = 1 \). This gives the final requirement that \( N_\Omega \mid N_p \). Gathering all the requirements together:

\[
F_d = N_o \quad (49)
\]

\[
N_d = 1 \quad (50)
\]

\[
F_o = 1 \quad (51)
\]

\[
N_p = kN_\Omega \forall \, k \in \mathbb{N} \quad (52)
\]
where \( N_0 \) is the desired number of orbits in the restricted Flower Constellation. In this case, \( N_0^* = N_\Omega F_d \) is the maximum number of satellites allowable.

Since there are \( F_d \) orbits, there are exactly \( N_\Omega \) permissible locations in which to place satellites per orbit. Curiously, \( F_n \) just so happens to equal \( N_d \) as was the requirement for a standard SCP arrayed about \( 2\pi \). Could we have started out with that assumption from the outset? The answer is no. The logic required to solve the restricted case is different and requires re-evaluating the governing equations from first principles resulting in a different solution path.

**Constructing Secondary Paths**

The developments of the previous sections have shown that there are specific requirements on the choice of parameters in order to ensure that a secondary path is formed. Figure 7 provides a sample of four constellations that have been generated using the concept of secondary closed paths.

![Fig. 7. Secondary closed paths can form on top of the relative orbit. These secondary closed paths rigidly rotate about the axis of symmetry while the relative path is fixed in the rotating reference frame. Note that the depictions of satellites are not to scale.](image)

**The Lone Star Constellation**

This section will examine how to create an example secondary closed path that is called the *Lone Star Constellation*. Looking at Figure 8(a), one can see that this constellation forms the shape of a five pointed star. This constellation closely resembles the star on flag of the State of Texas, hence the name. When viewing this constellation in motion, one would see the whole star spin about the axis of symmetry, which in this case is the spin axis of the Earth. Figure 8(b) and Figure 8(c) show the relative orbit and the ECI orbits, respectively, for this constellation. Based upon the developments previously described, the construction of the Lone Star Constellation is relatively simple. In this case, five secondary petals are desired to form on top of the relative path already in place. To effect this design, the value of \( F_d \) must be chosen appropriately based upon the value of \( \tau \) and for a particular choice of \( A \) and \( B \). Looking back at Theorem 4, one can immediately determine the appropriate values for \( A \) and \( B \) simply by noting that, if we want five points on the star, we require five loops in mean anomaly for one loop in RAAN about the Earth. This leads to \( A = 5 \) and \( B = 1 \).

However, \( A \) and \( B \) can be positive or negative. Therefore, we must resolve the question of appropriate sign. Generally, \( F_d \) is chosen to be positive because negative values for \( F_d \) simply reverse the sign on \( \Delta \Omega \) and \( \Delta M \) in Eqs. (3) and (4). Therefore, we choose \( A \) to be positive. The sign of \( B \) has physical connotations and can be determined by looking at how the five points of the star are created. When \( B = -1 \), we get the *Lone Star Constellation* pictured in Figure 8. However, when \( B = 1 \), the inverse pattern is created with five larger overlapping circles creating intersections where the vertices of the star would have been in the \( B = -1 \) case. Therefore, to obtain the desired constellation shape, we require

\[
F_d = 5N_d - N_p \tag{53}
\]

The values of \( N_d \) and \( N_p \) can be freely chosen to achieve a desired orbit period (i.e. a particular value of \( \tau \)). In order to make the points of the star fairly sharp, \( N_d = 23 \) and \( N_p = 38 \) were found to be acceptable, which leads to \( F_d = 5(23) - 38 = 77 \). This particular constellation is rather large, and smaller stars can be found by adjusting the values of \( N_d \) and \( N_p \).

The inclination of the *Lone Star Constellation* is set to be zero. This is done to maintain the sharp points of the constellation. Other values for the inclination will still yield a
A constellation with five points when viewed from the polar axis, but the points will be significantly rounded. For the equatorial \textit{Lone Star Constellation}, the choice of argument of perigee is arbitrary.

\textbf{Angular Velocity of a Secondary Path}

When all the admissible locations of a Flower Constellation are filled (especially when the number of these locations are many), the Flower Constellation dynamics reveals the shape of the relative trajectory by clearly showing the number of petals (i.e., the apogees of the relative trajectory). In this case the whole constellation appears to be rotating, as a rigid body, with the angular velocity of the planet, assuming that the orbit period is commensurate with the planet’s spin rate. Sometimes, however, the phasing does not allow us to fill out all the admissible locations, and it happens that the satellite distribution sequence comes back to the first position ($\Omega = 0$ and $M = 0$) before all the admissible locations are filled. When this happens, the Flower Constellation dynamics highlights the existence of Secondary Paths that have unexpected and beautiful shapes that are time invariant.$^3$

The immobility of the printed figure does not allow us to demonstrate the resulting complex shape-preserving dynamic. While complete Flower Constellations spin with a prescribed angular velocity (i.e. the same rate as that of the rotating reference frame), the spin rate of a secondary path should be quantified. Note that the angular velocity of a secondary path is \textit{apparent} and not real. That is to say, the apparent angular rotation is not a motion that can be described by any particular dynamical relationship but rather is an artifact of the mathematics that generates a Flower Constellation. In other words, the appearing angular rotation is not continuous but rather \textit{appears} continuous. Furthermore, the continuity is not discrete, as in the effect of the fast flow of photograms of motion pictures, because the satellite motion IS continuous. In effect, the angular motion arises because of a particular combination of the continuous motion of a satellite along its orbit and the discrete separation of contiguous orbits.

In general, SPs are made with single ($N_t = 1$) or multiple ($N_t > 1$) loops.$^3$ Figures (9) and (10) show a single and a four-loop SP, respectively. The angular velocity of a SP does not depend on $N_t$, but on the four integer parameters characterizing the loop(s): the number of apogees per loop, $N_{ap}$, the number of perigees per loop, $N_{pe}$, and two jumping-petal step parameters, $J_{ap}$ and $J_{pe}$. The jumping-petal step parameters indicate that the satellite moves from the petal apogee or perigee $k$ to the petal apogee or perigee $(k + J_{ap})$ or $(k + J_{pe})$, respectively. In each case, the petal apogees and perigees are counted counter clockwise and the jumping-step parameters are restricted to $0 \leq J_{ap} < N_{ap}$ and $0 \leq J_{pe} < N_{pe}$.

Figure (9) shows a SP having $N_{ap} = 10$ and $N_{pe} = 5$.

For a majority of SPs, we have $N_{ap} = N_{pe}$ and, for symmetry, $J_{ap} = J_{pe}$.

In the following, let us first consider the simple and common case of $N_{ap} = N_{pe}$, and identify as “petal” indifferently, petal-apogee or petal-perigee. In a SP, the time required from a satellite to move from a petal to the next petal is one orbit period. Therefore, in general, after one orbit period the satellite comes back to its initial position, but on a different petal of its loop.$^4$ After $N_{af}$ orbit periods, the satellite has visited all the petals of its loop. In the case the SP has one loop ($N_t = 1$), two petals ($N_{ap} = 2$), and a unitary jumping-petal step parameter ($J_{af} = 1$), then after $N_{af}$ orbit periods the SP is rotated by an angle $2\pi$ and, therefore, the SP angular velocity is $\omega = 2\pi/(N_{af}T)$. From these considerations, the intuitive solution for the general case, when the SP loop is characterized by any values for $N_{af}$ and $J_{af}$ can be determined.

There are two possible solutions associated with a clockwise

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig9.png}
\caption{A secondary path consisting of a single closed loop.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10.png}
\caption{A secondary path consisting of four distinct closed loops.}
\end{figure}

\footnotesize$^3$The shape of the loops in multiple-loop SPs are all identical. They are just rotated one from another by an angle $2\pi/N_t$.

\footnotesize$^4$If it comes back on the same petal, then the problem is trivial because the angular velocity of the SP is identical to that of the rotating reference frame.
or counter clockwise rotation
\[ \omega_{p,i} = \frac{2\pi}{T} \frac{N_{alt} - J_{alt}}{N_{alt}} \quad \text{and} \quad \omega_{p,i} = -\frac{2\pi}{T} \frac{J_{alt}}{N_{alt}} \] (54)

In the general case, when \( N_{alt} \neq N_{pel} \) (and \( J_{alt} \neq J_{pel} \)), there are an additional two angular velocities associated with the petal-perigee motion
\[ \omega_{p,i} = \frac{2\pi}{T} \frac{N_{pel} - J_{pel}}{N_{pel}} \quad \text{and} \quad \omega_{p,i} = -\frac{2\pi}{T} \frac{J_{pel}}{N_{pel}} \] (55)

Equivalent Satellite Distributions

The phasing theory of Flower Constellations generates a family of RAAN and mean anomaly angle pairs. Each pair of angles, called an element, is denoted \( \{d_i\}_{i \in I} \) where \( d_i \) is \( i^{th} \) member of the family and \( I \) is a nonempty index set. This family gives rise to the set \( \mathcal{D} = \{ d_i \mid i \in I \} \). Flower Constellation distribution sets are multisets. Multisets are set-like objects in which each element of the set is unique and the order of the elements is ignored. For instance, the multisets \( \{1,2,3\} \) and \( \{2,1,3\} \) are equivalent but \( \{1,1,3,2\} \) and \( \{2,3,1\} \) are different.

This follows from the physical meaning of the Flower Constellation distribution. When one places a satellite at a particular RAAN and mean anomaly position in a given orbit, it does not matter if that is the first satellite placed or the 3rd satellite placed - it is still the only satellite that is placed in that particular location.

Given an arbitrary Flower Constellation phasing distribution of finite length, one can choose to place the satellites in a particular order in a finite number of ways. In general, the number of Flower Constellation multisets of length \( k \) on \( n \) elements is called \( n \) multichoose \( k \), expressed as \( \binom{n}{k} \). This can be calculated as
\[ \binom{n}{k} = \binom{n+k-1}{k} = (n-1,k)! \] (56)

where \((n-1,k)!\) is a multinomial coefficient. Multinomial coefficients are calculated as
\[ \binom{n_1,n_2,\ldots,n_k}{!} = \frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\cdots n_k!} \] (57)

Therefore, the finite number of rearrangements of \( N \) elements of an arbitrary Flower Constellation distribution of length \( N \) is \( (2N-1)! \) where \( N \) is the number of satellites in the Flower Constellation. This number is important because it tells us something rather interesting. Until now, we have known that particular choices of the Flower Constellation parameters \( N_p \) and \( N_d \) will generate identical Flower Constellations (all other things being equal). With this development, we now can state that certain choices of \( F_n \) and \( F_d \) also produce identical Flower Constellations with all other things being equal.

This brings up the rather obvious question: Which choices of \( F_n \) and \( F_d \) will result in equivalent Flower Constellation multisets? Recognize that each choice of the phasing parameters really represents a choice of permutation of a fundamental Flower Constellation distribution, which we named previously a Contiguous Flower Constellation.

Families of Flower Constellations

Recall Eqs. (3) and (4) now written in a slightly different way.
\[ \Delta\Omega = -\frac{2\pi}{F_d} F_n = \Delta\Omega_c F_n \] (58)
\[ \Delta M_0 = \frac{2\pi}{F_d} F_n N_p = \Delta M_c F_n N_p = \Delta M_c F_s \] (59)

where it is clear that \( F_s = F_n N_p \in \{1,2,\ldots,F_d N_d - 1\} \). Note that in general, the quantity \( F_n N_p \) will not be able to represent all allowable values of \( F_s \). \( N_p \) is generally not a “free” parameter in this regard because \( N_p \), in conjunction with \( N_d \), specifies the orbit period. Therefore, our only option is to choose a value for \( F_n \). However, it is clear that the product of \( F_n \) and \( N_p \) can not reproduce the set of allowable values of \( F_s \) for arbitrary \( N_p \).

We desire to keep the phasing and shape considerations separate. Therefore, to encompass all possible values of \( F_s \), we must rewrite Equation (59). The simplest choice is a linear form
\[ F_s = F_n^* N_p + F_h F_d \] (60)
where \( F_h \in \{1,2,\ldots,N_d - 1\} \). Recall from the companion paper that \( F_h \) is a phasing step parameter. Here we must solve for the value of \( F_n^* = f(F_h) \) and obtain an equivalent Flower Constellation by setting \( F_n = F_n^* \).

Furthermore, one can deduce that
\[ \Delta M_c = -\frac{\Delta \Omega_c}{N_p} \] (61)

In Equation 59, there is a sequencing parameter for the mean anomaly that we have introduced in this paper as \( F_n \). In our previous work, the existence of this parameter was not appreciated. Therefore, we have modified our phasing relationships to account for the fact that one can choose the
ordering of the placement of satellites not only in terms of the right ascension of the ascending node but also in terms of the mean anomaly. However, this choice is not wholly independent as we shall demonstrate. \( F_\nu \) along with \( F_n \) represent quantities that describe families of Flower Constellations. The reason for this has to do with circular permutation theory.

A circular permutation is the number of ways in which one can arrange \( N \) distinct objects around a fixed closed planar path (e.g., a circle, ellipse, etc.). By fixed we mean to say that the closed path cannot be picked up off the plane and turned over (i.e., clockwise and counterclockwise have different meanings). Another way to envisage this is to imagine seating people around the dinner table. A circular permutation is the number of unique ways that you can arrange people around the table after the first person has chosen a seat. Please refer to Figures 12 and 13 for examples. As discussed earlier, a

Flower Constellation has precisely \( F_d \) orbits arrayed about \( 2\pi \). The first orbit is chosen to be placed at \( \Omega_0 \), typically equal to \( 0^\circ \). Therefore, according to circular permutation theory, there are \( P = (F_d - 1)! \) possible unique sequences of choices where one can place a satellite on subsequent orbits. The choice of sequence is controlled by the choice of \( F_n \).

When \( F_n = 1 \), then \( \Delta \Omega = \frac{2\pi}{F_d} = \Delta \Omega_c \), and satellites will be placed sequentially from one orbit to the next in a monotone

increasing manner (See Figure 14(a)). That is to say, \( \Omega_c \in \frac{2\pi}{F_d} \{1, 2, 3, \ldots, F_d - 1\} \). This is the first sufficient condition for what we shall call a Contiguous Flower Constellation. When \( F_n \neq 1 \), then the sequence of orbit choices will be shuffled and one might end up with, for example, \( \Omega \in \frac{2\pi}{F_d} \{1, F_d - 3, 2, F_d - 1, \ldots, F_d - 6\} \).

![Figure 12](image12.png)

Fig. 12. A graphical example of the allowable circular permutations for the \( N = 3 \) case. In this example, there are three admissible positions on a given orbit, and there are \((3 - 1)! = 2\) unique sequences that define the order of placement.

![Figure 13](image13.png)

Fig. 13. A graphical example of the allowable circular permutations for the \( N = 4 \) case. In this example, there are four admissible positions on a given orbit, and there are \((4 - 1)! = 6\) unique sequences that define the order of placement.

Note that in each case, the first value of RAAN will always be \( \Omega_0 \) or, in the set theory sequence, one. Additionally, this complete pattern will repeat \( N_d \leq N \) times until all satellites have been placed in unique RAAN and mean anomaly pairings.

Similarly, when \( F_n = 1 \), then

\[
\Delta M = \frac{2\pi}{F_d N_d} = \Delta M_c,
\]

and satellites will be placed in monotone increasing mean anomaly locations (See Figure 14(b)). That is to say,

\[
M_c \in \frac{2\pi}{F_d N_d} \{1, 2, 3, \ldots, F_d N_d - 1\}.
\]
This is a second sufficient condition to generate what we call a Contiguous Flower Constellation. Recognize though that a Flower Constellation can be contiguous with respect to either RAAN or mean anomaly but not both simultaneously.

The Contiguous Flower Constellation

Contiguous distributions have the property of distributing subsequent satellites in subsequent continuous orbits. This characteristic, as it will be clear later, allows us to evaluate the spin rate of the secondary path. A Contiguous Flower Constellation is generated by a family of elements \( \{ d_k \}_{k \in \mathbb{N}} \) where \( \mathbb{N} \) is the set of positive integers. Such a family is called a sequence and is a totally ordered set. A total order is a set where \( \Delta \Omega \) or \( \Delta \phi \) distributions that must be arrayed about the earth. We can write \( \Omega \) or \( \phi \) orbits that must be arrayed about the earth. We can write \( \Omega \) or \( \phi \) sequences and is a totally ordered set. A total order is a set where \( \Delta \Omega \) or \( \Delta \phi \) distributions that must be arrayed about the earth. We can write \( \Omega \) or \( \phi \) sequences and is a totally ordered set. A total order is a set where

\[ \forall x \in X \, (x \in X) \]

By establishing that \( F \equiv F_\Omega \), we can immediately choose \( Y = \Omega / \text{GCD}(\Omega) = \{ x/\text{GCD}(X) : x \in X \} \). Therefore, if \( \forall y \in Y_1 \equiv y \in Y_2 \) \( \Rightarrow Y_1 = Y_2 \) by the Axiom of Extensionality.

Note that \( F_\Omega = \text{GCD}(\Omega) \). \( F_\Omega \) has already been found and we can solve for \( F_n \). A similar analysis can be performed to extract the GCD of the mean anomaly distribution, which is equivalent to \( F_nN_p \). In the general case, \( F_n \neq F_\Omega \), and \( F_\Omega \neq F_\phi \). Therefore, by extracting the greatest common divisor from a given distribution, one can easily check to see if the fundamental distributions of each constellation constitute identical sets. If the sets are identical, then we can say that the two constellations are themselves identical.

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